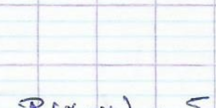
# Conditionnement - Indépendance

$X\ Y$ couple VA idem $(X_1 \dots X_d)$

| | VAD | VAC |
|---|---|---|
| Loi jointe de $(X,Y)$ | $P(X=x, Y=y)$ | densité jointe $f_{(XY)}(x,y)$<br>$P((X,Y)\in D) = \iint_D f_{(XY)}(x,y)\,dx\,dy$ |
| Loi conditionnelle de $X$ sachant $Y$ | $P(X=x \mid Y=y) = \dfrac{P(X=x, Y=y)}{P(Y=y)}$ | densité conditionnelle<br>$f_{X\mid Y=y}(x) = \dfrac{f_{(XY)}(x,y)}{f_Y(y)}$<br>$P(X\in A \mid Y=y) = \int_A f_{X\mid Y=y}(x)\,dx$ |
| Loi marginale de $X$ | $P(X=x) = \sum_y P(X=x, Y=y)$ | densité marginale<br>$f_X(x) = \int_{\mathbb{R}} f_{(XY)}(x,y)\,dy$ |
| Espérance | $E[g(X,Y)] = \sum_x\sum_y g(x,y)\,P(X=x, Y=y)$ | $E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y)\, f_{(XY)}(x,y)\,dx\,dy$ |
| Espérance condit de $X$ sachant $Y$ | $E[g(X,Y) \mid Y=y]$<br>$= \sum_x g(x,y)\, P(X=x \mid Y=y)$ | $\longrightarrow \int_{\mathbb{R}} g(x,y)\, f_{X\mid Y=y}(x)\,dx$ |

$$E[\overbrace{E[X \mid Y]}^{\psi(Y)}] = E[X]$$

$$= \sum_y E(X \mid Y=y)\, P(Y=y) \qquad \int E(X \mid Y=y)\, f_Y(y)\,dy$$




la "meilleure" fct de $X$ qui approche $Y$ est

$$\psi(X) = E(Y \mid X)$$

"meilleure" au sens du pdt scl $\langle T,U\rangle = E(TU)$

$$E(Y \mid X) = \underset{\varphi \text{ mesurable}}{\operatorname{argmin}}\ E[(Y - \varphi(X))^2]$$

Indép

$\mathbb{P}(X \in A, Y \in B)$

$= \mathbb{P}(X \in A)\, \mathbb{P}(X \in B) \quad \forall A, B \in \mathcal{B}(\mathbb{R})$

idem

$f_{(XY)}(x,y) = f_X(x)\, f_Y(y)$

$\mathbb{E}[g(X)\, h(Y)] = \mathbb{E}[g(X)]\, \mathbb{E}[h(Y)]$ indep $\Rightarrow \mathrm{Cov}(XY) = 0$

Fct génératrice $\forall z \in [-1\ 1]$

$G_X(z) = \mathbb{E}[z^X]$

fonct caractéristique $\forall z \in \mathbb{R}$

$\Psi_X(z) = \mathbb{E}[e^{izX}]$

$= \int e^{izx} f_X(x)\, dx$

$= \mathcal{F}(f_X)(z)$

$f = \frac{1}{2\pi} \int e^{-izx}\, \hat{f}$

$X, Y$ indep $\quad \Psi_{X+Y}(z) = \Psi_X(z)\, \Psi_Y(z)$

et $f_{X+Y}(t) = f_X(t) * f_Y(t)$

$= \int_{\mathbb{R}} f_X(t-y)\, f_Y(y)\, dy$

$F_X(x) = \mathbb{P}(X \leq x) \qquad f_X(x) = F_X'(x)$

$F_{X+Y}(t) = \mathbb{P}(X+Y \leq t) = \mathbb{P}((X,Y) \in \mathcal{D})$ avec $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \text{ tq } x+y \leq t\}$

$= \iint_{\mathcal{D}} \underbrace{f_{(X,Y)}(x,y)}_{= f_X(x)\, f_Y(y) \text{ car indep}}\, dx\, dy$

$= \int_{\mathbb{R}} \underbrace{\int_{-\infty}^{t-y} \left( f_X(x)\, dx \right)}_{=}\, f_Y(y)\, dy$

$= \int_{\mathbb{R}} F_X(t-y)\, f_Y(y)\, dy$

$f_{X+Y}(t) = \frac{\partial}{\partial t} F_{X+Y}(t) = \frac{\partial}{\partial t} \int_{\mathbb{R}} \int_{-\infty}^{t-y} f_X(x)\, dx\ f_Y(y)\, dy$

$= \frac{\partial}{\partial t} \int_{\mathbb{R}} F_X(t-y)\, f_Y(y)\, dy$

$= \int_{\mathbb{R}} \frac{\partial}{\partial t} F_X(t-y)\, f_Y(y)\, dy$

$$= f_X * f_Y(t)$$

Rappel $(U,V) = \varphi(X,Y)$ $\quad \varphi : \mathcal{C}^1$ difféo

$$\mathbb{P}((X,Y) \in \mathcal{D}) = \iint_{\mathcal{D}} f_{XY}(x,y)\, dx\, dy$$

$$= \iint_{\varphi(\mathcal{D})} f_{(U,V)}(u,v)\, du\, dv$$

$$f_{(U,V)}(u,v) = f_{(X,Y)}\left(\varphi^{-1}(u,v)\right) \times \left|\det \nabla \varphi^{-1}(u,v)\right|$$

$$\Psi = (\Psi_1, \dots \Psi_d) \qquad \nabla\Psi = \frac{\partial \Psi_i}{\partial x_j}_{1 \leq i, j \leq d}$$

$\Psi_1(x_1 \dots x_d)$
$\vdots$
$\Psi_d(x_1 \dots x_d)$