Corrigé de l'exam de Proba ENIT Mars 2016

Exercice 1

1) X suit la loi de Bernoulli $\mathcal{B}(\frac{1}{2})$.

$m = E(X) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$.

$\sigma^2 = E(X^2) = 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} = \frac{1}{2} \Rightarrow V(X) = E(X^2) - (E(X))^2 = \frac{1}{4}$.

2) a) $E(\overline{X_n}) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = E(X_1) = \frac{1}{2}$.

$$V(\overline{X_n}) = \frac{1}{n^2} \sum_{i=1}^{n} V(X_i) \quad \text{(car les } (X_i) \text{ indép.)}$$

$$= \frac{1}{n^2} \cdot n V(X_1) = \frac{1}{4n}.$$

b) * Par l'IBT :

$$P(|\overline{X_n} - m| > \alpha) \leq \frac{V(\overline{X_n})}{\alpha^2} = \frac{1}{4n\alpha^2}$$

$$\Rightarrow P(|\overline{X_n} - m| \leq 0.1) = 1 - P(\dots > 0.1)$$

$$\geq 1 - \frac{1}{4n(0.1)^2}$$

$$1 - \frac{1}{4n(0.1)^2} \geq 0{,}9 \iff n \geq \frac{1}{4 \cdot (0{,}1)^3} = 250.$$

* Par le TCL :

$$\frac{\overline{X_n} - E(\overline{X_n})}{\sqrt{Var(\overline{X_n})}} \underset{\approx}{\overset{Z \text{ de loi}}{}} \mathcal{N}(0,1)$$

$$P(|\overline{X_n} - m| \leq 0{,}1) = P\left(\frac{|\overline{X_n} - m|}{\sqrt{Var(\overline{X_n})}} \leq \frac{0{,}1}{\sqrt{Var(\overline{X_n})}} = 0{,}2\sqrt{n}\right)$$

$$\approx P(-0{,}2\sqrt{n} \leq Z \leq 0{,}2\sqrt{n}) = F_Z(0{,}2\sqrt{n}) - F_Z(-0{,}2\sqrt{n})$$

$$= F_Z(0{,}2\sqrt{n}) - (1 - F_Z(0{,}2\sqrt{n})) = 2F_Z(0{,}2\sqrt{n}) - 1.$$

$$2F_Z(0,2\sqrt{n}) - 1 \geq 0,9 \iff F_Z(0,2\sqrt{n}) \geq \frac{1,9}{2} = 0,95$$

$$\iff 0,2\sqrt{n} \geq F_Z^{-1}(0,95) \overset{(\text{Table})}{=} 1,64$$

$$\iff n \geq \left(\frac{1,64}{0,2}\right)^2 = 67,24 \Rightarrow n \geq 68.$$

Exercice 2.

1) $\int f_X = 1 \iff \beta \int_0^{+\infty} x^2 e^{-\alpha x}\, dx = 1$

$$\overset{(y=\alpha x)}{=} \frac{\beta}{\alpha^3}\int_0^{+\infty} y^2 e^{-y}\, dy = \frac{\beta}{\alpha^3}\Gamma(3) = \frac{2\beta}{\alpha^3}.$$

$$\Rightarrow \beta = \frac{\alpha^3}{2}.$$

2) $E(X) = 200$.

Or : $E(X) = \int_0^{+\infty} x f_X(x)\, dx = \int_0^{+\infty} \beta x^3 e^{-\alpha x}\, dx$

$$= \frac{\beta}{\alpha^4}\int_0^{+\infty} y^3 e^{-y}\, dy = \frac{\beta}{\alpha^4}\Gamma(4) = \frac{6\beta}{\alpha^4} = \frac{6\alpha^3/2}{\alpha^4} = \frac{3}{\alpha}.$$

Donc $200 = \frac{3}{\alpha} \Rightarrow \alpha = \frac{3}{200} = 0,015.$

3) Par identification : soit φ une fonction bornée (mesurable)

$$E(\varphi(Y)) = E(\varphi(\sqrt{X})) = \int_0^{+\infty} \varphi(\sqrt{x})\, \beta x^2 e^{-\alpha x}\, dx$$

$$\overset{(y=\sqrt{x})}{=} \int_0^{+\infty} \varphi(y)\, \beta y^4 e^{-\alpha y^2} . 2y\, dy$$

$$= \int_0^{+\infty} \varphi(y) . 2\beta y^5 e^{-\alpha y^2}\, dy.$$ D'où,

$$f_Y(y) = 2\beta y^5 e^{-\alpha y^2} \mathbb{1}_{y \geq 0}.$$

Exercice 3.

1) $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1 \iff c\int_0^1\int_0^1 y(1-x)\,dx\,dy = 1 \iff c = 4.$

2) * $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^1 4y(1-x)\,dy = 2(1-x), \ \forall x \in [0,1]$ (0 sinon)

* $f_Y(y) = \int f(x,y)\,dx = 2y, \ \forall y \in [0,1]$ (0 sinon).

* $E(X) = \int_{\mathbb{R}} x f_X(x)\,dx = \int_0^1 2x(1-x)\,dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{3}.$

* $\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f_X(x)\,dx = \left[\frac{2}{3}x^3 - \frac{2}{4}x^4\right]_0^1 = \frac{1}{6}$

$\Rightarrow \mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$

3) $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \Rightarrow X$ et Y indep

$\Rightarrow \mathrm{Cov}(X,Y) = 0.$

4) * $\mathbb{P}(X \leq \frac{2}{3}) = \int_{-\infty}^{2/3} f_X(x)\,dx = \int_0^{2/3} 2(1-x)\,dx = \frac{4}{3} - \frac{4}{9} = \frac{8}{9}.$

* $\mathbb{P}(X \leq Y) = \iint_{x \leq y} f_{(X,Y)}(x,y)\,dx\,dy = 4\int_0^1\int_0^y (1-x)\,dx\,y\,dy$

$= 4\int_0^1 y\left(y - \frac{y^2}{2}\right) dy = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}.$

Exercice 4.

1) $E = (A \cap D) \cup (\bar{A} \cap \bar{D})$

2) $\mathbb{P}(E) = \mathbb{P}(A \cap D) + \mathbb{P}(\bar{A} \cap \bar{D})$

$= \mathbb{P}(A|D)\mathbb{P}(D) + \mathbb{P}(\bar{A}|\bar{D})\mathbb{P}(\bar{D})$

$= (1 - 0{,}95) \times 0{,}01 + 0{,}02 \times 0{,}99 = 0{,}0203.$

3) $\mathbb{P}(D|A) = \frac{\mathbb{P}(A \cap D)}{\mathbb{P}(A)} = \frac{(1-0{,}95) \times 0{,}01}{\mathbb{P}(A|D)\mathbb{P}(D) + \underbrace{\mathbb{P}(A|\bar{D})}_{(1-\mathbb{P}(\bar{A}|\bar{D}))}\mathbb{P}(\bar{D})} = \frac{0{,}0005}{0{,}0005 + 0{,}98 \times 0{,}99}$

$= 0{,}00051.$