Corrigé de l'examen de Proba - Mars 2017.

Exercice 1.

1) D = "Pièce défectueuse"

$$\mathbb{P}(D) \underset{\text{"Probas. totales"}}{=} \mathbb{P}(D|A)\,\mathbb{P}(A) + \mathbb{P}(D|B)\,\mathbb{P}(B) + \mathbb{P}(D|C)\,\mathbb{P}(C)$$

$$= \frac{40}{100}\cdot\frac{2}{100} + \frac{4}{100}\cdot\frac{30}{100} + \frac{5}{100}\cdot\frac{30}{100}$$

$$= \frac{350}{10000} = \boxed{0{,}035}$$

2) $$\mathbb{P}(A|D) = \frac{\mathbb{P}(A\cap D)}{\mathbb{P}(D)} = \frac{\mathbb{P}(D|A)\cdot\mathbb{P}(A)}{\mathbb{P}(D)}$$

$$= \frac{\frac{2}{100}\cdot\frac{40}{100}}{\frac{350}{10000}} = \frac{8}{35} = \boxed{0{,}228}$$

Exercice 2.

1) $$\mathbb{P}(X < 20) = \frac{250}{400} = \mathbb{P}\Big(\overbrace{\frac{X-m}{\sigma}}^{Z} < \frac{20-m}{\sigma}\Big)$$

$$= F_Z\left(\frac{20-m}{\sigma}\right), \text{ où } Z \sim \mathcal{N}(0,1)$$

$$\mathbb{P}(X > 12) = \frac{380}{400} = 1 - F_Z\left(\frac{12-m}{\sigma}\right) = F_Z\left(\frac{m-12}{\sigma}\right)$$

$$\Rightarrow \begin{cases} \dfrac{20-m}{\sigma} = F_Z^{-1}\left(\dfrac{250}{400}\right) \overset{\text{table}}{\simeq} 0{,}32 \\ \dfrac{m-12}{\sigma} = F_Z^{-1}\left(\dfrac{380}{400}\right) \simeq 1{,}65 \end{cases} \Rightarrow \begin{cases} m = \boxed{18{,}6} \\ \sigma \simeq \boxed{4} \end{cases}$$

2) $\mathbb{P}(X>18) = \mathbb{P}\left(\frac{X-m}{\sigma} > \frac{18-m}{\sigma} = -\frac{0{,}6}{4} = -0{,}15\right)$

$= 1 - F_Z(-0{,}15) = F_Z(0{,}15) \overset{\text{table}}{=} \boxed{0{,}5596}$

3) $\mathbb{P}(X<18 \mid X>15) = \dfrac{\mathbb{P}(X<18, X>15)}{\mathbb{P}(X>15)}$

$$= \frac{\mathbb{P}\left(\frac{15-18{,}6}{4} < Z < \frac{18-18{,}6}{4}\right)}{\mathbb{P}\left(Z > \frac{15-18{,}6}{4}\right)}$$

$$= \frac{F_Z(-0{,}15) - F_Z\left(-\frac{3{,}6}{4}\right)}{1 - F_Z\left(-\frac{3{,}6}{4}\right)} = \frac{F_Z(0{,}9) - F_Z(0{,}15)}{F_Z(0{,}9)}.$$

$$= \frac{0{,}8159 - 0{,}5596}{0{,}8159} = \boxed{0{,}3141}.$$

Exercice 3.

1) $f \geq 0$ et $\int_{\mathbb{R}} f(x)\,dx = \int_0^{+\infty} x e^{-\frac{x^2}{2}}\,dx = \left[-e^{-\frac{x^2}{2}}\right]_0^{+\infty} = 1.$

2) $\mathbb{P}(X>1) = \int_1^{+\infty} f(x)\,dx = \left[-e^{-\frac{x^2}{2}}\right]_1^{+\infty} = \boxed{e^{-1/2}}.$

3) $\mathbb{E}(X) = \int_{\mathbb{R}} x f(x)\,dx = \int_0^{+\infty} x^2 e^{-\frac{x^2}{2}}\,dx \overset{..}{=} \boxed{\frac{\sqrt{2\pi}}{2}}$ (intégration par parties puis loi normale)

4) $Y = X^2$. Soit $\varphi : \mathbb{R} \to \mathbb{R}$ bornée mesurable.

$$\mathbb{E}[\varphi(Y)] = \mathbb{E}[\varphi(X^2)] = \int_{\mathbb{R}} \varphi(x^2) f_X(x)\, dx$$

$$= \int_0^{+\infty} \varphi(x^2)\, x e^{-\frac{x^2}{2}}\, dx \overset{y=x^2}{=} \int_0^{+\infty} \varphi(y) \sqrt{y}\, e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}}\, dy$$

$$= \int_{\mathbb{R}} \varphi(y)\, \underline{\frac{1}{2} e^{-\frac{y}{2}} \mathbb{1}_{y>0}}\, dy$$

$$\Rightarrow f_Y(y) = \frac{1}{2} e^{-y/2} \mathbb{1}_{y>0} \Rightarrow \text{loi } \mathcal{E}xp\left(\frac{1}{2}\right).$$

5) $\mathbb{E}(Y) = \int_0^{+\infty} y f_Y(y)\, dy = \dots \frac{1}{\frac{1}{2}} = 2.$

$\mathbb{E}(Y^2) = \int_0^{+\infty} y^2 f_Y(y)\, dy = \dots = 8.$

$\mathbb{V}ar(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 4.$

<u>Exercice 4</u>

1) $\iint f = 1 \Leftrightarrow C \int_0^{+\infty}\int_0^{+\infty} xy e^{-x^2} e^{-y^2} dx dy = 1$

$\Leftrightarrow \frac{C}{4} \left[-e^{-x^2}\right]_0^{+\infty} \left[-e^{-y^2}\right]_0^{+\infty} = 1 \Leftrightarrow \frac{C}{4} = 1 \Leftrightarrow$

$\Leftrightarrow \boxed{C = 4}.$

2) • $f_X(x) = \int_{\mathbb{R}} f_{(X,Y)}(x,y)\,dy$

$$= 4x e^{-x^2} \mathbb{1}_{x>0} \left[-\frac{1}{2} e^{-y^2}\right]_0^{+\infty}$$

$$= 2x e^{-x^2} \mathbb{1}_{x>0}.$$

• $f_Y(y) = 2y e^{-y^2} \mathbb{1}_{y>0}$ (car $f_{X,Y}$ symétrique en (x,y))

3) $f_{X,Y}(x,y) = f_X(x) f_Y(y) \Rightarrow X, Y$ <u>indép.</u>

4) Soit $\varphi : \mathbb{R} \to \mathbb{R}$ bornée mesurable.

$$\mathbb{E}[\varphi(Z)] = \mathbb{E}\left[\varphi\left(\sqrt{X^2+Y^2}\right)\right]$$

$$= \int_0^{+\infty}\int_0^{+\infty} \varphi\left(\sqrt{x^2+y^2}\right) 4xy\, e^{-(x^2+y^2)}\, dx\, dy$$

$$\overset{\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}}{=} \int_0^{+\infty}\int_0^{\frac{\pi}{2}} \varphi(r)\, 4r^2 \cos\theta \sin\theta\, e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{+\infty} \varphi(r) r^3 e^{-r^2} dr \underbrace{\int_0^{\pi/2} 2\sin(2\theta)\, d\theta}_{[-\cos 2\theta]_0^{\pi/2} = 2}$$

$$= \int_{\mathbb{R}} \varphi(r)\, \underline{2r^3 e^{-r^2} \mathbb{1}_{r>0}}\, dr$$

$$\Rightarrow f_Z(z) = 2z^3 e^{-z^2} \mathbb{1}_{z>0}.$$