(1)

Correction : exam Stat Mai 2018

Ex 1 :

1/ $$L(x_1, \dots, x_n, \alpha) = e^{-n\alpha} \frac{\alpha^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\log L(x_1, \dots, x_n, \alpha) = -n\alpha + \sum_{i=1}^{n} x_i \log \alpha + k.$$

$$\frac{\partial}{\partial \alpha} \log L(x_1, \dots, x_n, \alpha) = -n + \frac{\sum_{i=1}^{n} x_i}{\alpha} = 0$$

$$\iff \alpha = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\frac{\partial^2}{\partial \alpha^2} \log L(x_1, \dots, x_n, \alpha) = -\frac{\sum_{i=1}^{n} x_i}{\alpha^2} < 0$$

D'où $\hat{\alpha} = \overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

2/ $$E(\hat{\alpha}) = E(\overline{X}_n) = \frac{1}{n} \sum_{i=1}^{n} E(X_i)$$

Or $$E(X_i) = \sum_{n=0}^{+\infty} n P(X = n) = \sum_{n=0}^{+\infty} n e^{-\alpha} \frac{\alpha^n}{n!}$$

$$= \alpha e^{-\alpha} \sum_{n=1}^{+\infty} \frac{\alpha^{n-1}}{(n-1)!} = \underline{\alpha}$$

donc $\hat{\alpha}$ est un estimateur sans biais

2

$(X_i)_{1\le i\le n}$ sont iid d'après la loi des grands nombres

$\hat{\alpha} = \bar{X}_n$ converge vers $\mathbb{E}(X) = \alpha$

3/ D'après théorème TCL on a :

$$\frac{\bar{X}_n - \mathbb{E}(\bar{X}_n)}{\sqrt{\mathbb{V}(\bar{X}_n)}} \xrightarrow{\mathcal{L}oi} \mathcal{N}(0,1)$$

cherchons z_α tq $\mathbb{P}\left(\left|\frac{\bar{X}_n - \mathbb{E}(\bar{X}_n)}{\sqrt{\mathbb{V}(\bar{X}_n)}}\right| \le z_\alpha\right) = \frac{95}{100}$

$$2\Phi(z_\alpha) - 1 = 0,95 \Rightarrow \Phi(z_\alpha) = 0,975$$

donc $z_\alpha = 1,96$.

$$\mathbb{V}(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}(x_i) = \frac{1}{n}\mathbb{V}(X)$$

~~$\mathbb{E}(\alpha^2) = \sum n^2$~~

$$\mathbb{E}(X(X-1)) = \sum_{n=0}^{+\infty} n(n-1)\,\mathbb{P}(X=n)$$

$$= e^{-\alpha}\sum_{n=2}^{+\infty} n(n-1)\frac{\alpha^n}{n!} = e^{-\alpha}\sum_{n=2}^{+\infty}\frac{\alpha^n}{(n-2)!}$$

$$= \alpha^2.$$

③

$$\mathbb{P}\left(\alpha \in \left[\overline{X_n} - z_\alpha \sqrt{\mathbb{V}(\overline{X_n})}, \overline{X_n} + z_\alpha \sqrt{\mathbb{V}(\overline{X_n})}\right]\right) = ?$$ Sr

$$\mathbb{V}(\overline{X_n}) = \frac{1}{n}\mathbb{V}(X) = \frac{1}{n}\left(\alpha^2 + \alpha - \alpha^2\right) = \frac{\alpha}{n}.$$

On estime α par $\overline{X_n}$

On aura

$$I_\alpha = \left[\overline{X_n} - z_\alpha \sqrt{\frac{\overline{X_n}}{n}}, \overline{X_n} + z_\alpha \sqrt{\frac{\overline{X_n}}{n}}\right]..$$

Exercice 2 :

1/ $$\int_{\mathbb{R}} f(x)\,dx = \int_0^{+\infty} \frac{\lambda}{\theta^k} x^{k-1} e^{-x/\theta}\,dx = \frac{\lambda}{\theta^k}\int_0^{+\infty} (\theta y)^{k-1} e^{-y}\,\theta\,dy$$

$y = \frac{x}{\theta}$

$$= \frac{\lambda}{\theta^k}\theta^k \int_0^{+\infty} y^{k-1} e^{-y}\,dy = \lambda\,\Gamma(k) = \lambda(k-1)!$$

$= 1$

donc $$\boxed{\lambda = \frac{1}{(k-1)!}}$$

4

2) $L(x_1, \dots, x_n, \theta) = \left(\frac{\lambda}{\theta k}\right)^n \prod_{i=1}^{n} x_i^{k-1} e^{-\frac{1}{\theta}\sum x_i} \mathbb{1}_{(]0,+\infty[)^n}(x_1, \dots, x_n)$

pour $x_1, \dots, x_n \in]0, +\infty[$

$$\log L(x_1, \dots, x_n, \theta) = n \log \lambda - n \log(\theta) + \sum_{i=1}^{n} (k-1) \log x_i - \frac{1}{\theta} \sum_{i=1}^{n} x_i$$

$$\frac{\partial}{\partial \theta} \log L(x_1, \dots, x_n, \theta) = -\frac{nk}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i = 0$$

$$\iff \frac{nk}{\cancel{\theta}} = \frac{1}{\theta} \sum_{i=1}^{n} x_i \iff \theta = \frac{1}{nk} \sum_{i=1}^{n} x_i$$

$$\frac{\partial^2}{\partial \theta^2} \log L(x_1, \dots, x_n, \theta) = +\frac{nk}{\theta^2} - \frac{2}{\theta^3} \sum_{i=1}^{n} x_i = \frac{1}{\theta^2} \left[nk - \frac{2}{\theta} \sum_{i=1}^{n} x_i \right]$$

$$\left.\frac{\partial^2 \log L}{\partial \theta^2}\right|_{\theta = \frac{\bar{x}_n}{k}} = -\frac{nk}{\theta^2} < 0$$

Donc $\hat{\theta} = \frac{\bar{X}_n}{k}$

3/ $\mathbb{E}(\hat{\theta}) = \frac{1}{nk}\sum_{i=1}^{n}\mathbb{E}(X_i) = \frac{\mathbb{E}(X)}{k} = \hat{\theta}$ (5)

Cou $\mathbb{E}(X) = \int_0^{+\infty} x \frac{\lambda}{\theta^k} x^{k-1} e^{-\frac{x}{\theta}} dx, \; y = \frac{x}{\theta}$

$= \int_0^{+\infty} \theta y \frac{\lambda}{\theta^k} (\theta y)^{k-1} e^{-y} dy\, \theta$

$= \int_0^{+\infty} \theta \lambda y^k e^{-y} dy = \theta \Gamma(k+1) \lambda = \theta k!$

$= \underline{\underline{k\theta}}$

$\mathbb{E}(X^2) = \int_0^{+\infty} \theta^2 \lambda y^{k+1} e^{-y} dy = \Gamma(k+2)\, \theta^2 \lambda$

$= (k+1)!\, \theta^2 \lambda$

$= (k+1) k \theta^2$

$\mathbb{V}(X) = k(k+1)\theta^2 - k^2\theta^2 = k\theta^2$

$\cancel{\mathbb{V}(\hat{\theta})}$ $\mathbb{V}(\overline{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}(X_i) = \frac{k\theta^2}{n}$ $\Rightarrow \mathbb{V}(\hat{\theta}) = \mathbb{V}\left(\frac{\overline{X}_n}{k}\right) = \frac{\theta^2}{kn}$

4/ $\mathbb{E}(\hat{\theta}) = \theta \Rightarrow \hat{\theta}$ est un estimateur sans biais

et d'après la loi des grands nombres

$$\hat{\theta} = \frac{\overline{X}_n}{k} \longrightarrow \frac{\mathbb{E}(X)}{k} = \theta \Rightarrow \hat{\theta} \text{ sans biais}$$

Cela veut dire que $\hat{\theta}$ est un estimateur convergent (1ère L.G.N).

5/ $$I_n(\theta) = \mathbb{V}\left(\frac{\partial}{\partial\theta} \log L\right) = \mathbb{V}\left(\frac{-nk}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{\theta^4}\sum_{i=1}^{n} \mathbb{V}(X_i) = \frac{1}{\theta^4}\, nk\theta^2 = \frac{nk}{\theta^2}.$$

$$\Rightarrow I_n(\theta) = \frac{1}{\mathbb{V}(\hat{\theta})} \xRightarrow{\text{et sans biais}} \hat{\theta} \text{ optimal (efficace).}$$

Exercice 3

1) (X_i) iid $\overset{\text{par T.C.L}}{\Longrightarrow}$ $$\frac{\overline{X_n} - \mathbb{E}(\overline{X_n})}{\sqrt{\mathbb{V}(\overline{X_n})}} \xrightarrow[n\to+\infty]{} \mathcal{N}(0,1)$$

$$\Rightarrow \overline{X_n} \sim \mathcal{N}\left(\mathbb{E}(\overline{X_n}), \mathbb{V}(\overline{X_n})\right)$$

$$\mathbb{E}(\overline{X_n}) = \frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}(X_i)}_{p} = p.$$

$$\mathbb{V}(\overline{X_n}) = \frac{\mathbb{V}(X_i)}{n} = \frac{p(1-p)}{n}.$$

2) $$\left[\overline{X_n} - t_\alpha\sqrt{\frac{S_n^2}{n}}, \ \overline{X_n} + t_\alpha\sqrt{\frac{S_n^2}{n}}\right],$$

avec
$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X_n})^2$$
$$t_\alpha = F^{-1}_{\underset{\text{Student}}{St(n-1)}}\left(1-\frac{\alpha}{2}\right)$$

$$\simeq F^{-1}_{\mathcal{N}(0,1)}\left(1-\frac{\alpha}{2}\right).$$

$$= F^{-1}_{\mathcal{N}(0,1)}\left(1-\frac{0,98}{2}\right)$$

$$= F^{-1}_{\mathcal{N}(0,1)}(0,51) \overset{\text{Table}}{=} 0,02 \text{ (ou } 0,03).$$

(et si on regard $F^{-1}_{St(99)}(0,51) \rightsquigarrow$ la plus proche dans la table Student est 0,126.