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Correction Examen Statistique session principale

Ex 1 :

1) $\mathbb{E}(\overline{X_n}) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(X_k) = m \longrightarrow$ linéaire

$\mathbb{V}(\overline{X_n}) = \frac{1}{n^2} \sum_{k=1}^{n} \mathbb{V}(X_k) = \frac{\sigma^2}{n}$ (indépendance)

2) $Z = \frac{\overline{X_n} - m}{\frac{\sigma}{\sqrt{n}}} = \mathcal{N}(0,1)$

Determiner a_α tq $\mathbb{P}(|Z| \leq a_\alpha) = 90\% = 1-\alpha$

$$2\Phi_{N(0,1)}(a_\alpha) - 1 = \frac{90}{100} \Rightarrow \Phi_{N(0,1)}(a_\alpha) = \frac{0,9+1}{2} = 0,95$$

$$\Rightarrow \boxed{a_\alpha = 1,65}$$

$$\Rightarrow I = \left[\overline{X_n} - 1,65\frac{\sigma}{\sqrt{n}}, \overline{X_n} + 1,65\frac{\sigma}{\sqrt{n}}\right]$$

3) $l(I) = 2 \times 1,65 \frac{\sigma}{\sqrt{n}} = 0,001$

$$\Leftrightarrow n = \left(\frac{3,3}{0,001} \times 6\right)^2 = \left(\frac{3,3 \times 0,05}{0,001}\right)^2$$

$$= \left(\frac{10^{-3}}{10^{-3}} \times 165\right)^2$$

$$= (165)^2 = 27225$$

②

Problème :

1/ $f_\alpha \geq 0$, f_α est continue sur $]-\infty, 1[$ et sur $]1, +\infty[$

$\lim_{x \to 1^+} f_\alpha(x) = \alpha = f_\alpha(1)$ et $\lim_{x \to 1^-} f_\alpha(x) = 0$

f_α continue par morceaux sur $\mathbb{R}$

et $\int_{\mathbb{R}} f_\alpha(x)\,dx = \int_1^{+\infty} \frac{\alpha}{x^{\alpha+1}}\,dx = \alpha \left[-\frac{1}{\alpha} x^{-\alpha-1+1} \right]_1^{+\infty}$

$= \frac{\alpha}{\alpha} = 1$

2/ $\mathbb{E}(X_i) = \int_{\mathbb{R}} x f_\alpha(x)\,dx = \int_1^{+\infty} \alpha \frac{x}{x^{\alpha+1}}\,dx = \alpha \int_1^{+\infty} \frac{1}{x^\alpha}\,dx$

$= \alpha \left[\frac{1}{1-\alpha} x^{1-\alpha} \right]_1^{+\infty} = \begin{cases} \frac{\alpha}{\alpha-1} & \text{si } \alpha > 1 \\ \infty & \text{si } 0 < \alpha \leq 1 \end{cases}$

$\mathbb{E}(X_i^2) = \int_{\mathbb{R}} x^2 f_\alpha(x)\,dx = \alpha \left[\frac{1}{2-\alpha} x^{2-\alpha} \right]_1^{+\infty} = \begin{cases} \frac{\alpha}{\alpha-2} & \text{si } \alpha > 2 \\ \infty & \text{si } 0 < \alpha \end{cases}$

3) ~~$\tilde{\alpha}_n = \overline{X}_n$~~

On estime $\mathbb{E}(X_1) = \frac{\alpha}{\alpha-1}$ par $\overline{X}_n$

donc $\frac{\tilde{\alpha}_n}{\hat{\alpha}_n - 1} = \overline{X}_n \Rightarrow \hat{\alpha}_n = \overline{X}_n(\hat{\alpha}_n - 1)$

$\Rightarrow \hat{\alpha}_n (1 - \overline{X}_n) = -\overline{X}_n$

$$\Rightarrow \boxed{\tilde{\alpha}_n = \frac{\overline{X}_n}{\overline{X}_n - 1}}$$

b) par la loi des grand nombre $\underline{\alpha > 1}$

$\overline{X}_n \longrightarrow \mathbb{E}(X_1)$ (convergente)

par la continuité $\Rightarrow \tilde{\alpha}_n \longrightarrow \frac{\mathbb{E}(X_1)}{\mathbb{E}(X_1) - 1} = \frac{\frac{\alpha}{\alpha-1}}{\frac{\alpha}{\alpha-1} - 1} = \frac{\frac{\alpha}{\cancel{\alpha-1}}}{\frac{\alpha - \alpha + 1}{\cancel{\alpha-1}}} = \alpha$

④

7/

$$L(x_1, \dots, x_n, \alpha) = \prod_{i=1}^{n} \frac{\alpha}{x_i^{\alpha+1}} \mathbb{1}_{[1,+\infty[}(x_i)$$

$$= \frac{\alpha^n}{\left(\prod_{i=1}^{n} x_i\right)^{\alpha+1}} \mathbb{1}_{\{1 \leq \max(x_i)\}}$$

8/ $x_i \in [1, +\infty[$

$$\text{Log}\, L(x_1, \dots, x_n, \alpha) = n \,\text{Log}\, \alpha - (\alpha+1) \sum_{i=1}^{n} \text{Log}\, x_i$$

$$\frac{\partial}{\partial \alpha} \text{Log}(x_1, \dots, x_n, \alpha) = \frac{n}{\alpha} - \sum_{i=1}^{n} \text{Log}\, x_i = 0$$

$$\Longleftrightarrow \alpha = \frac{n}{\sum_{i=1}^{n} \text{Log}\, x_i} \text{ et comme } \frac{\partial^2}{\partial \alpha^2} \text{Log}\, L(x_1, \dots, x_n) = \frac{-n}{\alpha^2} < 0$$

$$\Longrightarrow \hat{\alpha}_n = \frac{n}{\sum_{k=1}^{n} \text{Log}\, X_k}$$

7/ 5

$$F_Y(y) = \mathbb{P}(\operatorname{Lng} X_1 \leq y)$$

$$= \mathbb{P}(X_1 \leq e^y) = \begin{cases} 0 & \text{si } e^y \leq 1 \Leftrightarrow y \leq 0 \\ \displaystyle\int_1^{e^y} \frac{\alpha}{x^{\alpha+1}}\,dx & \text{si } y \geq 0 \end{cases}$$

$$\int_1^{e^y} \frac{\alpha}{x^{\alpha+1}}\,dx = \left[\frac{-1}{\alpha} x^{-\alpha}\right]_1^{e^y} = \frac{1 - e^{-\alpha y}}{\alpha}$$

$$\Rightarrow f_Y(y) = \begin{cases} e^{-\alpha y} & \text{si } y > 0 \\ 0 & \text{si } \underline{\text{non}} \end{cases}$$

8/ $$\mathbb{E}[\operatorname{lng} X_1] = \mathbb{E}[Y] = \int_0^{+\infty} y e^{-\alpha y}\,dy$$

$$= \frac{1}{\alpha}$$

$$\mathbb{E}[\operatorname{Lng}^2 X_1] = \mathbb{E}(Y^2) = \int_0^{+\infty} y^2 e^{-\alpha y}\,dy$$

$$= \frac{2}{\alpha^2}$$

9°/ posons $Y_k = \log X_k$ (6)

$$\Rightarrow \bar{Y}_n = \frac{1}{n}\sum_{k=1}^{n} Y_k = \frac{1}{n}\sum_{k=1}^{n} \log X_k$$

D'après la loi de grand nombre

$\bar{Y}_n$ Converge vers $\mathbb{E}(Y) = \frac{1}{\alpha}$

$$\Rightarrow \frac{n}{\sum \log X_k} \text{ Converge vers } \alpha.$$

10°/ $\cancel{\mathbb{V}\left(\hat{\alpha}_n\right) = \mathbb{V}\left(\frac{1}{\bar{Y}_n}\right)} =$

$$I_n(\alpha) = \mathbb{E}\left[-\frac{\partial^2 \log L(x_1, \dots, x_n, \alpha)}{\partial \alpha^2}\right]$$

$$= \mathbb{E}\left[-\left(-\frac{n}{\alpha^2}\right)\right] = \frac{n}{\alpha^2}.$$

$\cancel{11°/ \ \mathbb{P}\left(\left|\frac{\hat{\alpha}_n - \mathbb{E}(\hat{\alpha}_n)}{\sqrt{\mathbb{V}(\hat{\alpha}_n)}}\right| \leq a_\alpha\right) = 1 - \alpha}$

11/ posons $Z = \dfrac{\hat{\lambda}_n - \lambda}{\frac{\hat{\lambda}_n}{\sqrt{n}}} \sim \mathcal{N}(0,1)$ (7)

Déterminer c tq

$$\mathbb{P}(|Z| \leq c) = 0,95$$

$$2\Phi_{\mathcal{N}(0,1)}(c) - 1 = 0,95 \Rightarrow$$

$$\boxed{c = 1,96}$$

$$\mathbb{P}\left(-c\frac{\hat{\lambda}_n}{\sqrt{n}} + \hat{\lambda}_n \leq \lambda \leq \hat{\lambda}_n + c\frac{\hat{\lambda}_n}{\sqrt{n}}\right) = 95\%$$

$$I_F = \left[\hat{\lambda}_n - 1,96\frac{\hat{\lambda}_n}{\sqrt{n}},\ \hat{\lambda}_n + \frac{1,96}{\sqrt{n}}\hat{\lambda}_n\right]$$