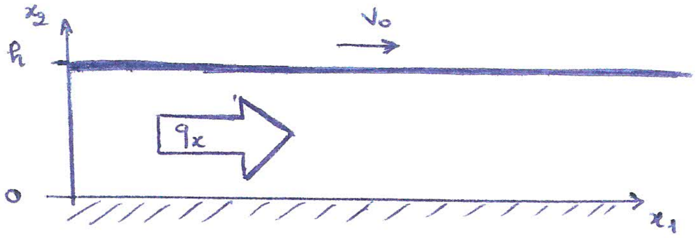
Série 3

Exercice 1:



var. d'Euler x_1, x_2, x_3, t

Sys. d'écoulement $\begin{cases} \text{permanent} & \Rightarrow \dfrac{\partial}{\partial t} = 0 \\ \text{bidim. } (x_1, x_2) & \Rightarrow \dfrac{\partial}{\partial x_3} = 0 \quad \text{et} \quad v_3 = 0 \\ \text{parallèle} & \Rightarrow v_2 = 0 \end{cases}$

1./ éq. de Navier-Stokes:

$$\begin{cases} \dfrac{\partial v_i}{\partial x_i} = 0 \\ \dfrac{\partial v_i}{\partial t} + v_j \dfrac{\partial v_i}{\partial x_j} = -\dfrac{1}{\rho} \dfrac{\partial P}{\partial x_i} + \nu \dfrac{\partial^2 v_i}{\partial x_j^2} + g_i \end{cases}$$

dans le sys: $\dfrac{\partial v_1}{\partial x_1} = 0 \Rightarrow v_1 = v_1(x_2)$

la sym donne $\vec{v} = v_1(x_1, x_2)\vec{e}_1$
la cons. de la masse donne $\vec{v} = v_1(x_2)$

d'où
$$\begin{cases} -\dfrac{1}{\rho} \dfrac{\partial P}{\partial x_1} + \nu \dfrac{d^2 v_1}{dx_2^2} = 0 \\ -\dfrac{1}{\rho} \dfrac{\partial P}{\partial x_2} + g_2 = 0 \\ -\dfrac{1}{\rho} \dfrac{\partial P}{\partial x_3} = 0 \end{cases}$$

2./ $\dfrac{\partial v_1}{\partial x_1} = 0$ donc l'écoulement est établi

3./ Cdts aux limites: en $x_2 = 0$ et en $x_2 = h$ $\begin{cases} v_1(x_2 = 0) = 0 \\ v_1(x_2 = h) = v_0 \end{cases}$

3.a/ Si $\dfrac{dP}{dx_1} = 0$ et $v_0 \neq 0$

on a $-\dfrac{1}{\rho} \dfrac{\partial P}{\partial x_3} = 0 \Rightarrow P = P(x_1, x_2)$

$-\dfrac{1}{\rho} \dfrac{\partial P}{\partial x_2} + g_2 = 0 \Rightarrow P = \rho g x_2 + C(x_1)$

$\Rightarrow \dfrac{\partial P}{\partial x_1} = C'(x_1) = 0$ par hyp. $\Rightarrow C(x_1) = \text{cste} = C$

$\Rightarrow \boxed{P = \rho g x_2 + C}$

$-\dfrac{1}{\rho} \cancel{\dfrac{\partial P}{\partial x_1}} + \nu \dfrac{d^2 v_1}{dx_1^2} = 0 \Rightarrow \dfrac{d^2 v_1}{dx_1^2} = 0$

$\Rightarrow \dfrac{dv_1}{dx_2} = \alpha \Rightarrow v_1 = \alpha x_2 + \beta$

$\Rightarrow \boxed{v_1 = \dfrac{v_0}{h} x_2}$

3.b/ Si $\frac{dP}{dx_1} = -\Gamma < 0$ et $v_0 = 0$

$$\frac{\Gamma}{\rho} + \nu \frac{d^2 v_1}{dx_2} = 0$$

C.L : $v_1(x_2=0) = v_1(x_2=h) = 0$

$$\Rightarrow \frac{d^2 v_1}{dx_2} = -\left(\frac{\Gamma}{\nu\rho}\right) = -\left(\frac{\Gamma}{\mu}\right)$$ ← viscosité dynamique

$$\frac{d v_1}{dx_2} = -\left(\frac{\Gamma}{\nu\rho}\right) x_2 + C_1$$

$$v_1 = -\left(\frac{\Gamma}{2\mu}\right) x_2^2 + C_1 x_2 + C_2$$

$v_1(x_2=0) = 0 \Rightarrow C_2 = 0$

$v_1(x_2=h) = 0 \Rightarrow -\frac{\Gamma}{2\mu} h^2 + C_1 h = 0 \Rightarrow C_1 = \frac{\Gamma}{2\mu} h$

d'où $$\boxed{v_1 = -\left(\frac{\Gamma}{2\mu}\right) x_2^2 + \frac{\Gamma}{2\mu} h\, x_2}$$

• Taux de cisaillement:

*/ $\tau = 2\mu D \Rightarrow \tau = \mu \left[\vec{\nabla}\vec{v} + {}^t(\vec{\nabla}\vec{v})\right]$ $\boxed{d\vec{F}_{cis} = \tau.\vec{n}}$

$$\Rightarrow \tau_{ij} = \mu \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]$$

$$\tau_{12} = \tau_{21} = \mu \frac{dv_1}{dx_2}$$

$$\tau_{12} = -\frac{\Gamma h^2}{2\mu}\left[\frac{2x_2}{h^2} - \frac{1}{h}\right]$$

5. Q = [Débit] $= L^3 T^{-1} = (L T^{-1}) L^2$

volume / temps vitesse x surface (section de la conduite)

$$Q = \int_{x_2=0}^{x_2=h} v_1\, dx_2 \Rightarrow Q = -\left(\frac{\Gamma}{6\mu}\right) h^3 + \frac{\Gamma}{4\mu} h^3 = \frac{\Gamma}{12\mu} h^3$$

$Q_{réel} = b\,Q$ ← largeur

$$\boxed{v_{moy} = \frac{Q}{h}}$$

6./ $Q_{inj} = bQ \Rightarrow Q = \frac{Q_{inj}}{b} = \frac{0,1}{2} = 0,05\ m^2.s^{-1}$

$$\Gamma = \frac{dP}{dx_2} = \frac{\Delta P}{L} = \frac{P_{inj} - P_{atm}}{L}$$

Exercice 2 : Ecoulement laminaire d'un fluide visqueux dans une conduite cylindrique

écoulement :
- permanent $\Rightarrow \frac{\partial}{\partial t} = 0$
- axisym $\Rightarrow \frac{\partial}{\partial \theta} = 0$ et $v_\theta = 0$ $\Rightarrow \vec{v} = v_z(r,z)\vec{u_z}$
- laminaire $\Rightarrow v_r = 0$
- établi $\Rightarrow \frac{\partial v}{\partial z} = 0$

$\vec{v} = v_z(r)\vec{u_z}$

1./ éq. de cons de la masse $\Rightarrow \frac{\partial v_z}{\partial z} = 0$ $\Rightarrow$ l'écoulement est parallèle

2./ Navier - Stokes $\Rightarrow$

$$\begin{cases} -\frac{1}{\rho}\frac{\partial P}{\partial r} = 0 \\ -\frac{1}{\rho}\frac{\partial P}{\partial \theta} = 0 \\ \underbrace{-\frac{1}{\rho}\frac{dP}{dz}}_{-\Gamma} + \nu \underbrace{\Delta v_z}_{= \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right]} = 0 \end{cases}$$

$\Rightarrow P = P(z)$

3./ $\frac{\partial P}{\partial x} = 0$ (en fait $x \leftrightarrow z$)

4./ $\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] = \frac{-\Gamma}{\nu\rho} = -\frac{\Gamma}{\mu}$

C.L $v_z(r=R) = 0$

$r \to 0$ vitesse infinie $\Rightarrow C_1 = 0$

$\left[r\frac{\partial v_z}{\partial r}\right] = -\frac{\Gamma}{2\mu}r^2 + C_1$

$\frac{\partial v_z}{\partial r} = -\frac{\Gamma}{2\mu}r + \frac{C_1}{r}$ $\Rightarrow v_z(r) = -\frac{\Gamma}{4\mu}r^2 + C_1 \ln\left(\frac{r}{R}\right) + K$

or $v_z(r=R) = 0$

$\Rightarrow K = \frac{\Gamma}{4\mu}R^2$

$v_z = -\frac{\Gamma R^2}{4\mu}\left[\left(\frac{r}{R}\right)^2 - 1\right]$

le sens de l'écoulement est contraire au gradient de pression

$v_{moy} = \frac{Q}{S}$ où $S = \pi R^2$

$Re = \frac{UD}{\nu}$ (U : vitesse moyenne ; ν : viscosité)

$\nu_{eau} = 10^{-6}\ m^2 s^{-1}$

$Re_{eau} = 2000$

$UD = 2.10^{-3}\ m^2 s^{-1}$

$U = 2.10^{-2}\ m/s$

$U = 2\ cm/s$

$Re_{critique} = 2200$

si on dépasse Re_c alors on perd les hyp. initiales

Eq. N.S $\quad g_z = -g\sin\alpha \quad g>0$: terme qui s'ajoute

$\varphi = ---$

trouver Γ ---

$$\Gamma = \frac{P_{inj} - P_{atm}}{L} \quad \text{où} \quad L = \frac{H}{\sin\alpha}$$

Exercice 3 :

sys d'écoulement : $\begin{cases} \text{permanent} & \frac{\partial}{\partial t} = 0 \\ \text{bidim } (x,y) : & \frac{\partial}{\partial z} = 0 \text{ et } \quad w = 0 \\ \text{établi} & \frac{\partial}{\partial x} = 0 \end{cases}$

vitesse $= (u, v, w)$

Eq. de Navier_Stokes

$$\begin{cases} \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{établi}} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{\text{bidim}} = 0 \ (1) \iff \frac{\partial v_i}{\partial x_i} = 0 \\ \cancel{\frac{\partial v_i}{\partial t}} + \cancel{v_j \frac{\partial v_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j^2} + g_i \ (2) \end{cases}$$

(2) :
$$\begin{cases} 0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{d^2 u}{dy^2} + g\sin\alpha & (i) \text{ Proj sur } x \\ 0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - g\cos\alpha & (ii) \text{ Proj sur } y \\ 0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} & (iii) \text{ Proj sur } z \end{cases}$$

(iii) $\Rightarrow P(x,y,z) = P(x,y)$

(ii) $\Rightarrow P = -(\rho g\cos\alpha)y + P_0(x)$

or à $y = h \Rightarrow P(y=h, x) = -(\rho g\cos\alpha)h + P_0(x) = P_{atm}$

$$P = P(y) = \rho g\cos\alpha\,(h - y) + P_{atm}$$

(i) $\Rightarrow 0 = 0 + \nu \frac{d^2u}{dy^2} + g\sin\alpha$

C.L $\begin{cases} \text{au radier} \quad u(y=0) = 0 \\ \text{à la surface libre} \quad \tau_{xy} = \tau_{yx} = \mu \left.\frac{du}{dy}\right|_{y=h} = 0 \end{cases}$

$$\frac{d^2u}{dy^2} = \frac{-g\sin\alpha}{\nu}$$

$$\frac{du}{dy} = -\left(g\frac{\sin\alpha}{\nu}\right)y + c_1$$

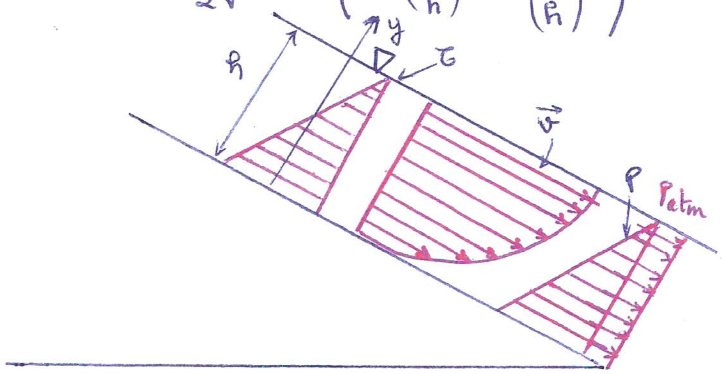
$$u = -\left(g\,\frac{\sin\alpha}{2\nu}\right)y^2 + c_1\,y + c_2$$

$$u(y=0) = 0 \quad \Rightarrow \quad c_2 = 0$$

$$\left.\frac{du}{dy}\right|_{y=h} = 0 \quad \Rightarrow \quad \left(\frac{-g\sin\alpha}{\nu}\right)h + c_1 = 0 \quad \Rightarrow \quad c_1 = \frac{g\sin\alpha}{\nu}\,h$$

$$u(y) = -g\left(\frac{\sin\alpha}{2\nu}\right)h^2\left[\left(\frac{y}{h}\right)^2 - 2\left(\frac{y}{h}\right)\right]$$

$$u(y) = g\,\frac{\sin\alpha}{2\nu}\,h^2\left(2\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right)$$



$$\tau_{xy} = \tau_{yx} = \mu\,\frac{du}{dy} = \mu\,\frac{g\sin\alpha\,h^2}{2\nu}\left[\frac{\cancel{2}}{h} - \frac{\cancel{2}y}{h^2}\right] \qquad \nu = \frac{\mu}{\rho}$$

$$= \rho g\sin\alpha\,h\left(1 - \frac{y}{h}\right)$$

$$\tau_{radier} = \tau|_{y=0} = \rho g\sin\alpha\,h$$